Math 126 Problem 17: Insights from the Edge-incidence Matrix

Spring 2027

Submission Instructions

Please upload your work to Gradescope by **11:59pm on Monday, April 12, 2027 (Pacific time).**

- You are encouraged to discuss ideas and work with your classmates. However, you *must acknowledge* your collaborators at the top of each solution on which you collaborated with others and you *must write* your solutions independently.
- Your solutions to theory questions must be written legibly, or typeset in LaTeX or markdown.
- I recommend that you write your solutions in a Python notebook using Google Colab.
- You should submit your solutions as a *single PDF* via the assignment on Gradescope.

Grading: The point of the problem set is for *you* to learn. To this end, I hope to disincentivize the use of LLMs by *not* grading your work for correctness. Instead, you will grade your own work by comparing it to my solutions. This self-grade is due the Friday *after* the problem set is due, also on Gradescope.
Original notebook: Google Colab.

Insights from the Edge-incidence Matrix

Let the number of edges m be greater than the number of nodes n. The edge-incidence matrix $\mathbf{B} \in \mathbb{R}^{m \times n}$ is an incredibly powerful tool for analyzing graph topology. In the notes, we saw that the Laplacian can be written as $\mathbf{L} = \mathbf{B}^\top \mathbf{B}$. In this problem, we will look at the solutions to $\mathbf{Bx} = \mathbf{0}$ and $\mathbf{B}^\top \mathbf{y} = \mathbf{0}$ to uncover fundamental structural properties of the graph $G = (V, E)$.

Part A: Node Vectors and Connected Components

Consider a node vector $\mathbf{x} \in \mathbb{R}^n$ such that $\mathbf{Bx} = \mathbf{0}$. Observe that $\mathbf{Bx} = \mathbf{0}$ if and only if $\mathbf{Lx} = \mathbf{0}$.

Use the definition of $\mathbf{B}$ to show that if $\mathbf{Bx} = \mathbf{0}$, then $x_i = x_j$ for every edge $(i, j) \in E$. Use this observation to formally prove that if the graph has k connected components, the multiplicity of the eigenvalue 0 of $\mathbf{L}$ is exactly k.

Part B: Edge Vectors and Cycles

Now consider an edge vector $\mathbf{y} \in \mathbb{R}^m$ such that $\mathbf{B}^\top \mathbf{y} = \mathbf{0}$. We can think of each entry y_e as a "flow" assigned to the edge e, respecting the arbitrary direction assigned to e by the definition of $\mathbf{B}$. First, interpret the equation $\mathbf{B}^\top \mathbf{y} = \mathbf{0}$ at a specific node $v \in V$. What physical conservation law does this equation represent for the "flows" passing through v?

Next, recall from linear algebra that the number of independent solutions to a system of equations is equal to the number of variables minus the rank of the matrix. By Part A, we know that the rank of $\mathbf{B}$ is $n - k$ where k is the number of connected components. Using this fact, write down the number of independent solutions to $\mathbf{B}^\top \mathbf{y} = \mathbf{0}$ in terms of m, n, and k. Based on this formula and your physical interpretation of the flows, prove that there exists a non-zero solution $\mathbf{y}$ if and only if the graph G contains at least one cycle.