Math 126 Problem 8: Sphere in a Box

Spring 2027

Submission Instructions

Please upload your work to Gradescope by **11:59pm on Monday, February 22, 2027 (Pacific time).**

- You are encouraged to discuss ideas and work with your classmates. However, you *must acknowledge* your collaborators at the top of each solution on which you collaborated with others and you *must write* your solutions independently.
- Your solutions to theory questions must be written legibly, or typeset in LaTeX or markdown.
- I recommend that you write your solutions in a Python notebook using Google Colab.
- You should submit your solutions as a *single PDF* via the assignment on Gradescope.

Grading: The point of the problem set is for *you* to learn. To this end, I hope to disincentivize the use of LLMs by *not* grading your work for correctness. Instead, you will grade your own work by comparing it to my solutions. This self-grade is due the Friday *after* the problem set is due, also on Gradescope.
Original notebook: Google Colab.

Sphere in a Box

Consider a d-dimensional hypercube centered at the origin with side length 4 (so it spans $[-2, 2]$ in each dimension). Place 2^d unit spheres (radius $r = 1$) into the corners of the hypercube. Each sphere is centered at a coordinate $(\pm 1, \pm 1, \ldots, \pm 1)$.

Now, place a smaller central sphere at the origin so that it exactly touches all 2^d corner spheres. Using the Pythagorean theorem, calculate the distance from the origin to the center of a corner sphere. Deduce the radius R_d of this central sphere.

Here is the weird part. We know the distance from the origin to the face of the hypercube is 2 (since the side length is 4). Find the dimension d where $R_d = 2$. (At this point, the central sphere is tangent to the outside walls of the box). What happens when $d > 9$?

Describe the shape of the central sphere relative to the hypercube for extremely large d. How can a sphere enclosed by other spheres poke outside the box containing them?